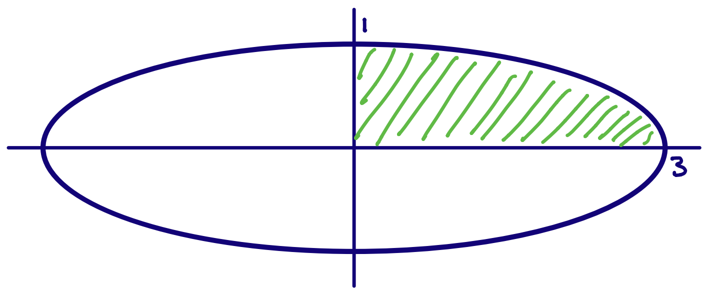
7.3 TRIGONOMETRIC SUBSTITUTION

Example.- AREA OF AN ELLIPSE



$$\left(\frac{x}{3}\right)^2 + \left(\frac{y}{1}\right)^2 = 1 \quad \text{(EQ. OF ELLIPSE)}$$

$$\frac{1}{9}x^2 + y^2 = 1 \Rightarrow y^2 = 1 - \frac{1}{9}x^2 = \frac{1}{9}(9 - x^2)$$

$$\Rightarrow y = \pm \frac{1}{3}\sqrt{9 - x^2}$$

$$\boxed{\frac{1}{4}\text{AREA} = \int_0^3 \frac{1}{3}\sqrt{9 - x^2}\, dx}$$

IDEA: INVERSE SUBSTITUTION

$x = 3\sin\theta \qquad x = 0 \longrightarrow \theta = 0$

$dx = 3\cos\theta\, d\theta \qquad x = 3 \longrightarrow \theta = \frac{\pi}{2}$

$$\frac{1}{4}\text{ AREA} = \int_0^{\pi/2} \frac{1}{3}\sqrt{9 - 9\sin^2\theta}\,(3\cos\theta)\,d\theta$$

$$= \int_0^{\pi/2} \sqrt{3^2(1-\sin^2\theta)}\,\cos\theta\,d\theta \;=\; \left|\begin{array}{l}\cos^2\theta \\ \quad \shortparallel \\ 1-\sin^2\theta\end{array}\right.$$

$$= \int_0^{\pi/2} \sqrt{3^2\cos^2\theta}\,\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 3\cos\theta\cos\theta\,d\theta = \int_0^{\pi/2} 3\cos^2\theta\,d\theta =$$

HALF ANGLE

$$= \int_0^{\pi/2} 3\cdot\frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$= \frac{3}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2}$$

$$= \frac{3}{2}\left(\frac{\pi}{2} + \frac{1}{2}\sin\pi - 0 - \frac{1}{2}\sin 0\right)$$

$$= \frac{3\pi}{4}$$

$$\boxed{\text{AREA} = 3\pi}$$