Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a\sin\theta,\quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta,\quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

$$\int \frac{1}{x^2\sqrt{9-x^2}}\,dx$$

$x = 3\sin\theta$

$dx = 3\cos\theta\,d\theta$

$$= \int \frac{1}{3^2\sin^2\theta\sqrt{3^2 - 3^2\sin^2\theta}}\,3\cos\theta\,d\theta$$

$$= \int \frac{1}{3^2\sin^2\theta\cdot\cancel{3}\sqrt{1-\sin^2\theta}}\,\cancel{3}\cos\theta\,d\theta$$

$$= \frac{1}{9}\int \frac{1}{\sin^2\theta\sqrt{1-\sin^2\theta}}\cos\theta\,d\theta$$

$1-\sin^2\theta = \cos^2\theta$

$\sqrt{1-\sin^2\theta} = \cos\theta$

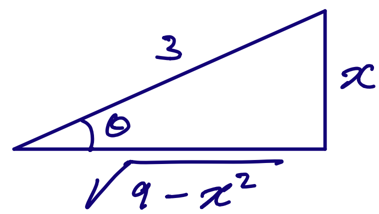
$$= \frac{1}{9}\int \frac{1}{\sin^2\theta\cdot\cancel{\cos\theta}}\cdot\cancel{\cos\theta}\,d\theta$$

$$= \frac{1}{9}\int \csc^2\theta\,d\theta$$

$$= \frac{1}{9}(-\cot\theta) + C$$

$x = 3\sin\theta$ $\quad$ $\sin\theta = \frac{x}{3}$

$\theta = \arcsin\frac{x}{3}$



$\cot\theta = \frac{\sqrt{9-x^2}}{x}$

$$= -\frac{1}{9}\frac{\sqrt{9-x^2}}{x} + C$$

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$$\int \frac{1}{x^2\sqrt{x^2+4}}\,dx \qquad \begin{aligned} x &= 2\tan\theta \\ dx &= 2\sec^2\theta\,d\theta \end{aligned}$$

$$= \int \frac{\cancel{2}\sec^2\theta}{4\tan^2\theta\sqrt{\cancel{4}\tan^2\theta+\cancel{4}}}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\sqrt{\tan^2\theta+1}}\,d\theta = \frac{1}{4}\int \frac{\sec^2\theta}{\tan^2\theta\,\sec\theta}\,d\theta$$

$$= \frac{1}{4}\int \frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{4}\int \frac{1/\cos\theta}{\sin^2\theta/\cos^2\theta}\,d\theta$$

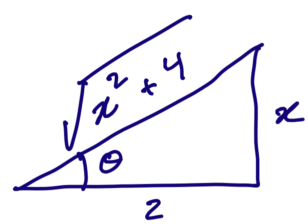
$$= \frac{1}{4}\int \frac{\cos\theta}{\sin^2\theta}\,d\theta \qquad \begin{aligned} u &= \sin\theta \\ du &= \cos\theta\,d\theta \end{aligned}$$

$$= \frac{1}{4}\int \frac{1}{u^2}\,du = \frac{1}{4}\left(-\frac{1}{u}\right) + C$$

$$= -\frac{1}{4}\,\frac{1}{\sin\theta} + C$$

$$= -\frac{1}{4}\,\frac{\sqrt{x^2+4}}{x} + C$$

$$x = 2\tan\theta \qquad \tan\theta = \frac{x}{2}$$



$$\sin\theta = \frac{x}{\sqrt{x^2+4}}$$

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$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx \qquad \left|\begin{array}{l} x = a\sec\theta \\ dx = a\sec\theta\tan\theta\,d\theta \end{array}\right. \qquad \boxed{a>0}$$

$$= \int \frac{\cancel{a}\sec\theta\tan\theta}{\sqrt{\cancel{a^2}\sec^2\theta - \cancel{a^2}_{1}}}\,d\theta = \int \frac{\sec\theta\tan\theta}{\sqrt{\sec^2\theta - 1}}\,d\theta$$

$$= \int \frac{\sec\theta\,\cancel{\tan\theta}}{\cancel{\tan\theta}}\,d\theta = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\right| + C_1$$

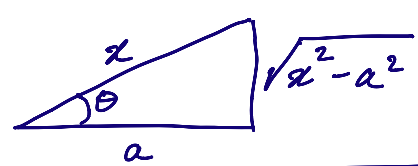
$$= \ln\left|x + \sqrt{x^2-a^2}\right| - \ln a + C_1$$

$$= \ln\left|x + \sqrt{x^2-a^2}\right| + C'$$

$$x = a\sec\theta$$

$$\sec\theta = \frac{x}{a}$$

$$\cos\theta = \frac{a}{x}$$



$$\tan\theta = \frac{\sqrt{x^2-a^2}}{a}$$