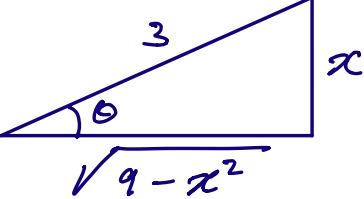


### Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

$$\begin{aligned}
 & \int \frac{1}{x^2 \sqrt{9-x^2}} dx \quad \left| \begin{array}{l} x = 3 \sin \theta \\ dx = 3 \cos \theta d\theta \end{array} \right. \\
 &= \int \frac{1}{3^2 \sin^2 \theta \sqrt{3^2 - 3^2 \sin^2 \theta}} 3 \cos \theta d\theta \\
 &= \int \frac{1}{3^2 \sin^2 \theta \cdot 3 \sqrt{1 - \sin^2 \theta}} \cancel{3 \cos \theta} d\theta \\
 &= \frac{1}{9} \int \frac{1}{\sin^2 \theta \sqrt{1 - \sin^2 \theta}} \cos \theta d\theta \quad \left| \begin{array}{l} 1 - \sin^2 \theta = \cos^2 \theta \\ \sqrt{1 - \sin^2 \theta} = \cos \theta \end{array} \right. \\
 &= \frac{1}{9} \int \frac{1}{\sin^2 \theta \cdot \cancel{\cos \theta}} \cdot \cancel{\cos \theta} d\theta \\
 &= \frac{1}{9} \int \csc^2 \theta d\theta
 \end{aligned}$$

$$= \frac{1}{9} (-\cot \theta) + C$$

$x = 3 \sin \theta \quad \sin \theta = \frac{x}{3}$   
 $\theta = \arcsin \frac{x}{3}$   
  
 $\cot \theta = \frac{\sqrt{9-x^2}}{x}$

$$= -\frac{1}{9} \frac{\sqrt{9-x^2}}{x} + C$$

### Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

$$\int \frac{1}{x^2 \sqrt{x^2+4}} dx \quad x = z \tan \theta \\ dx = z \sec^2 \theta d\theta$$

$$= \int \frac{z \sec^2 \theta}{4 \tan^2 \theta \sqrt{4 \tan^2 \theta + 4}} d\theta$$

$$= \frac{1}{4} \int \frac{\sec^2 \theta}{\tan^2 \theta \sqrt{\tan^2 \theta + 1}} d\theta = \frac{1}{4} \int \frac{\sec^2 \theta}{\tan^2 \theta \sec \theta} d\theta$$

$$= \frac{1}{4} \int \frac{\sec \theta}{\tan^2 \theta} d\theta = \frac{1}{4} \int \frac{1/\cos \theta}{\sin^2 \theta / \cos^2 \theta} d\theta$$

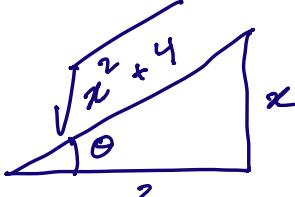
$$= \frac{1}{4} \int \frac{\cos \theta}{\sin^2 \theta} d\theta \quad \left| \begin{array}{l} u = \sin \theta \\ du = \cos \theta d\theta \end{array} \right.$$

$$= \frac{1}{4} \int \frac{1}{u^2} du = \frac{1}{4} \left( -\frac{1}{u} \right) + C$$

$$= -\frac{1}{4} \frac{1}{\sin \theta} + C \quad x = z \tan \theta \quad \tan \theta = \frac{x}{z}$$

$$= -\frac{1}{4} \frac{\sqrt{x^2+4}}{x} + C$$

(Red line)



$$\sin \theta = \frac{x}{\sqrt{x^2+4}}$$

### Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin \theta, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}$	$1 - \sin^2 \theta = \cos^2 \theta$
$\sqrt{a^2 + x^2}$	$x = a \tan \theta, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2 \theta = \sec^2 \theta$
$\sqrt{x^2 - a^2}$	$x = a \sec \theta, \quad 0 \leq \theta < \frac{\pi}{2} \text{ or } \pi \leq \theta < \frac{3\pi}{2}$	$\sec^2 \theta - 1 = \tan^2 \theta$

$$\begin{aligned}
 & \int \frac{1}{\sqrt{x^2 - a^2}} dx \quad \left| \begin{array}{l} x = a \sec \theta \\ dx = a \sec \theta \tan \theta d\theta \end{array} \right. \quad a > 0 \\
 &= \int \frac{1 \cancel{\sec \theta \tan \theta}}{\sqrt{a^2 \sec^2 \theta - a^2}} d\theta = \int \frac{\sec \theta \tan \theta}{\sqrt{\sec^2 \theta - 1}} d\theta \\
 &= \int \frac{\sec \theta \cancel{\tan \theta}}{\cancel{\tan \theta}} d\theta = \int \sec \theta d\theta \\
 &= \ln |\sec \theta + \tan \theta| + C \\
 &= \ln \left| \frac{x}{a} + \frac{\sqrt{x^2 - a^2}}{a} \right| + C \\
 &= \ln |x + \sqrt{x^2 - a^2}| - \ln a + C \\
 &= \underline{\ln |x + \sqrt{x^2 - a^2}| + C'} \quad \left| \begin{array}{l} x = a \sec \theta \\ \sec \theta = \frac{x}{a} \\ \cos \theta = \frac{a}{x} \\ \text{Diagram: A right triangle with hypotenuse } x, \text{ adjacent side } a, \text{ and angle } \theta. \\ \tan \theta = \frac{\sqrt{x^2 - a^2}}{a} \end{array} \right.
 \end{aligned}$$