$\boxed{7.4}$ PARTIAL FRACTIONS

GOAL: Integrals of RATIONAL functions

Recall: RATIONAL func.: $\dfrac{P(x)}{Q(x)}$

where both $P(x)$ and $Q(x)$ are POLYNOMIALS

e.g. - $\dfrac{3x^2 - 2x + x^7}{x + x^5 - x^2}$ RATIONAL

$\sqrt{x}$, $\ln x$, e^x NOT RATIONAL

$\dfrac{x^2 + x^{1/2}}{x + x^{-1}}$ NOT RATIONAL

IDEA: WRITE $\dfrac{P(x)}{Q(x)}$ IN TERMS OF SIMPLER FRACTIONS

Example:

$$\int \frac{x+5}{x^2+x-2}\,dx$$

Note :

$$\frac{2}{x-1} - \frac{1}{x+2} = \frac{2(x+2) - 1\cdot(x-1)}{(x-1)(x+2)}$$

$$= \frac{x+5}{x^2+x-2}$$

Then :

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \left(\frac{2}{x-1} - \frac{1}{x+2}\right) dx$$

$$= 2\ln|x-1| - \ln|x+2| + C$$

CASE 1 : DISTINCT LINEAR FACTORS

$$\frac{px+q}{(\alpha x+a)(\beta x+b)} = \frac{A}{\alpha x+a} + \frac{B}{\beta x+b}$$

$$\frac{px^2+qx+r}{(\alpha x+a)(\beta x+b)(\gamma x+c)} = \frac{A}{\alpha x+a} + \frac{B}{\beta x+b} + \frac{C}{\gamma x+c}$$

SIMILAR FOR MORE FACTORS

$$\int \frac{x-7}{x^2-x-12}\,dx$$

FACTOR DENOMINATOR:

$$x^2-x-12 = (x+3)(x-4)$$

$$\frac{x-7}{(x+3)(x-4)} = \frac{A}{x+3} + \frac{B}{x-4}$$

FIND: A and B

$$= \frac{A(x-4) + B(x+3)}{(x+3)(x-4)}$$

$$x-7 = A(x-4) + B(x+3)$$

$x=4$ $\longrightarrow$ $\underbrace{4-7}_{=-3} = A\underbrace{(4-4)}_{=0} + B\underbrace{(4+3)}_{=7}$

$$-3 = B\cdot 7$$

$$B = -\frac{3}{7}$$

$x=-3$ $\longrightarrow$ $\underbrace{-3-7}_{=-10} = A\underbrace{(-3-4)}_{=-7} + B\underbrace{(-3+3)}_{=0}$

$$-10 = A\cdot(-7)$$

$$A = \frac{10}{7}$$

$$\int \frac{x-7}{x^2-x-12}\,dx = \int\left(\frac{10/7}{x+3} + \frac{-3/7}{x-4}\right)dx$$

$$= \frac{10}{7}\ln|x+3| - \frac{3}{7}\ln|x-4| + C$$

$\boxed{\text{CASE 1 : DISTINCT LINEAR FACTORS}}$

$$\frac{px+q}{(\alpha x+a)(\beta x+b)} = \frac{A}{\alpha x+a} + \frac{B}{\beta x+b}$$

$$\frac{px^2+qx+r}{(\alpha x+a)(\beta x+b)(\gamma x+c)} = \frac{A}{\alpha x+a} + \frac{B}{\beta x+b} + \frac{C}{\gamma x+c}$$

SIMILAR FOR MORE FACTORS

$\boxed{\text{CASE 2 : REPEATED LINEAR FACTORS}}$

$$\frac{R(x)}{(\alpha x+a)(\beta x+b)^2} = \frac{A}{\alpha x+a} + \frac{B}{\beta x+b} + \frac{C}{(\beta x+b)^2}$$

$$\frac{R(x)}{(\alpha x+a)^2(\beta x+b)(\gamma x+c)^3} =$$

$$= \frac{A}{\alpha x+a} + \frac{B}{(\alpha x+a)^2} + \frac{C}{\beta x+b} + \frac{D}{\gamma x+c} + \frac{E}{(\gamma x+c)^2} + \frac{F}{(\gamma x+c)^3}$$

SIMILAR FOR MORE FACTORS

$$\int \frac{5x-3}{x^2(x+3)}\,dx$$

$$\frac{5x-3}{x^2(x+3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+3}$$

$$= \frac{Ax(x+3) + B(x+3) + Cx^2}{x^2(x+3)}$$

$$5x-3 = A\,x(x+3) + B\,(x+3) + C\,x^2$$

$$\xrightarrow{x=0} \quad -3 = A\cdot 0 + B\cdot 3 + C\cdot 0 \qquad B=-1$$

$$\xrightarrow{x=-3} \quad -18 = A\cdot 0 + B\cdot 0 + C\,(-3)^2 \qquad C=-2$$

$$\xrightarrow{x=1} \quad 2 = A\cdot 4 + B\cdot 4 + C\cdot 1$$

$$2 = A\cdot 4 - 4 - 2 = 4A - 6$$

$$8 = 4A \qquad A = 2$$

$$\int \frac{5x-3}{x^2(x+3)}\,dx = \int \left(\frac{2}{x} - \frac{1}{x^2} - \frac{2}{x+3}\right) dx$$

$$= 2\ln|x| + \frac{1}{x} - 2\ln|x+3| + C$$

CASE 2 : REPEATED LINEAR FACTORS

$$\frac{R(x)}{(\alpha x+a)^2(\beta x+b)(\gamma x+c)^3} =$$

$$= \frac{A}{\alpha x+a} + \frac{B}{(\alpha x+a)^2} + \frac{C}{\beta x+b} + \frac{D}{\gamma x+c} + \frac{E}{(\gamma x+c)^2} + \frac{F}{(\gamma x+c)^3}$$

CASE 3,4 : QUADRATIC FACTORS

$$\frac{R(x)}{(x+a)(x+b)^3(x^2+cx+d)(x^2+ex+f)^2} =$$

$$= \frac{A}{x+a} + \frac{B}{x+b} + \frac{C}{(x+b)^2} + \frac{D}{(x+b)^3} +$$

$$+ \frac{Ex+F}{x^2+cx+d} + \frac{Gx+H}{x^2+ex+f} + \frac{Ix+J}{(x^2+ex+f)^2}$$

SIMILAR FOR MORE FACTORS

$$\int \frac{3x^2+2}{(x-1)(x^2+4)}\,dx$$

$$\frac{3x^2+2}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

$$\frac{3x^2+2}{(x-1)(x^2+4)} = \frac{A(x^2+4) + (Bx+C)(x-1)}{(x-1)(x^2+4)}$$

$$3x^2+2 = A(x^2+4) + (Bx+C)(x-1)$$

$$3x^2+2 = \underline{A}\,x^2 + \underline{4A} + \underline{B}\,x^2 - \underline{B}\,x + \underline{C}\,x - \underline{C}$$

$$\underline{3}\,x^2 + \underline{0}\cdot x + \underline{2} = \underline{(A+B)}\,x^2 + \underline{(C-B)}\,x + \underline{(4A-C)}$$

$A+B=3 \longrightarrow A+B=3$ *

$C-B=0 \longrightarrow B=C$ *

$4A-C=2$

$\rightarrow 4A-B=2$

$\rightarrow 5A=5$

$\downarrow$

$A=1$

$A=1$

* $A+B=3$

$\rightarrow 1+B=3 \rightarrow B=2 \xrightarrow{*} C=2$

$$\int \frac{3x^2+2}{(x-1)(x^2+4)}\,dx = \int \frac{1}{x-1}\,dx + \int \frac{2x+2}{x^2+4}\,dx$$

$$= \int \frac{1}{x-1}\,dx + \int \frac{2x}{x^2+4}\,dx + \int \frac{2}{x^2+4}\,dx$$

$$= \ln|x-1| + \ln|x^2+4| + \arctan\left(\frac{x}{2}\right) + C$$

we used: $\displaystyle\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$