

$$\int \frac{1}{x^2 - x^4} dx$$

$$\begin{aligned}\frac{1}{x^2 - x^4} &= \frac{1}{x^2(1-x^2)} = \frac{1}{x^2(1-x)(1+x)} \\ &= \frac{x(1-x)(1+x)}{x(1-x)(1+x)} \cdot \frac{A}{x} = \frac{Ax(1-x)(1+x)}{x^2(1-x)(1+x)} \\ &= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{1-x} + \frac{D}{1+x}\end{aligned}$$

$$= \frac{Ax(1-x)(1+x) + B(1-x)(1+x) + Cx^2(1+x) + Dx^2(1-x)}{x^2(1-x)(1+x)}$$

$$\begin{aligned}1 &= A x(1-x)(1+x) \\ &+ B(1-x)(1+x) \\ &+ Cx^2(1+x) \\ &+ Dx^2(1-x)\end{aligned}$$

$$\begin{aligned}x = -1 \quad 1 &= D \cdot (-1)^2 (1-(-1)) \\ D &= \frac{1}{2}\end{aligned}$$

$$\begin{aligned}x = 1 \quad 1 &= C \cdot 1^2 (1+1) \\ C &= \frac{1}{2}\end{aligned}$$

$$\underline{x=0} \quad 1 = B(1-0)(1+0) \quad \boxed{B=1}$$

$$\underline{x=2} \quad 1 = A(-6) + B(-3) + C(2) + D(-8)$$

$$1 = -6A - 3 + 6 - 4 = -6A - 1$$

$$2 = -6A \quad \boxed{A = -\frac{1}{3}}$$

$$\int \left( \frac{-1/3}{x} + \frac{1}{x^2} + \frac{1/2}{1-x} + \frac{1/2}{1+x} \right) dx$$

$$= -\frac{1}{3} \ln|x| - \frac{1}{x} - \frac{1}{2} \ln|1-x| + \frac{1}{2} \ln|1+x| + C$$


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$$\int \frac{x^5 + x^3 + 2x^2 + 1}{x^4 + x^2} dx = I$$

$$\begin{array}{r} x \\ \hline x^4 + x^2 \quad \left[ \begin{array}{r} x^5 + x^3 + 2x^2 + 1 \\ x^5 + x^3 \\ \hline 2x^2 + 1 \end{array} \right] \end{array}$$

$$\frac{x^5 + x^3 + 2x^2 + 1}{x^4 + x^2} = x + \frac{2x^2 + 1}{x^4 + x^2} = x + \frac{2x^2 + 1}{x^2(x^2 + 1)}$$

IRREDUCIBLE  
QUADRATIC

$$\frac{2x^2 + 1}{x^2(x^2 + 1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx + D}{x^2 + 1}$$

$$2x^2 + 1 = A x(x^2 + 1) + B(x^2 + 1) + (Cx + D)x^2$$

$$\begin{aligned}
 \frac{2x^2+1}{x^2(x^2+1)} &= \frac{(x^2)+(x^2+1)}{x^2(x^2+1)} = \\
 &= \frac{\cancel{x^2}}{\cancel{x^2}(x^2+1)} + \frac{\cancel{x^2+1}}{\cancel{x^2}(x^2+1)} \\
 &= \frac{1}{x^2+1} + \frac{1}{x^2} \quad \begin{array}{l} A=0 \\ B=1 \\ C=0 \\ D=1 \end{array}
 \end{aligned}$$

$$\begin{aligned}
 I &= \int \left( x + \frac{1}{x^2+1} + \frac{1}{x^2} \right) dx \\
 &= \frac{1}{2}x^2 + \arctan(x) - \frac{1}{x} + C
 \end{aligned}$$


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$$\int \frac{x^4 - 3x^3 + 4x^2 - 2x + 1}{x^2 - 2x + 2} dx \quad \begin{array}{l} \text{TRY THIS} \\ \text{AT HOME} \end{array}$$


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$$\begin{aligned}
 I_2 &= \int \frac{1}{x^2 - 2x + 2} dx \quad \begin{array}{l} \text{COMPLETE} \\ \text{THE SQUARE} \end{array} \\
 &\quad \text{IRRED.} \\
 &\quad \text{QUAD.}
 \end{aligned}$$

$$I_2 = \int \frac{1}{\underbrace{(x^2 - 2x + 1)}_{= (x-1)^2} + 1} dx$$

$$= \int \frac{1}{(x-1)^2 + 1} dx$$

$$\left. \begin{aligned} u &= x-1 \\ du &= dx \end{aligned} \right|$$

$$= \int \frac{1}{u^2 + 1} du$$

$$= \arctan(u) + C$$

$$= \underline{\arctan(x-1) + C}$$