$\boxed{7.8}$ **Improper Integrals**

$\int_a^b f(x)\,dx$ is called IMPROPER if

$\boxed{f(x) \text{ is DISCONTINUOUS in } [a,b]}$ OR $\boxed{a = -\infty}$ OR $\boxed{b = +\infty}$

1 Definition of an Improper Integral of Type 1

(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

$$\int_a^\infty f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

provided this limit exists (as a finite number).

(b) If $\int_t^b f(x)\,dx$ exists for every number $t \leqslant b$, then

$$\int_{-\infty}^b f(x)\,dx = \lim_{t\to-\infty} \int_t^b f(x)\,dx$$

provided this limit exists (as a finite number).

The improper integrals $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^b f(x)\,dx$ are called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If both $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^a f(x)\,dx$ are convergent, then we define

$$\int_{-\infty}^\infty f(x)\,dx = \int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx$$

In part (c) any real number a can be used (see Exercise 76).

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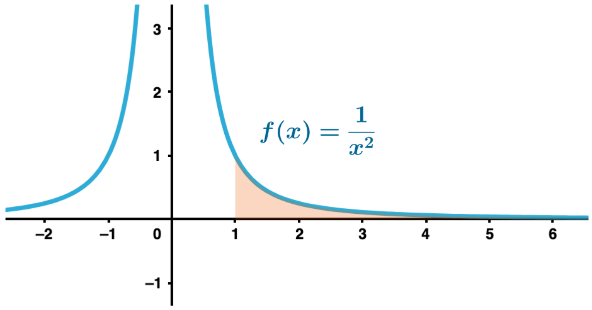


(a) If $\int_a^t f(x)\,dx$ exists for every number $t \geqslant a$, then

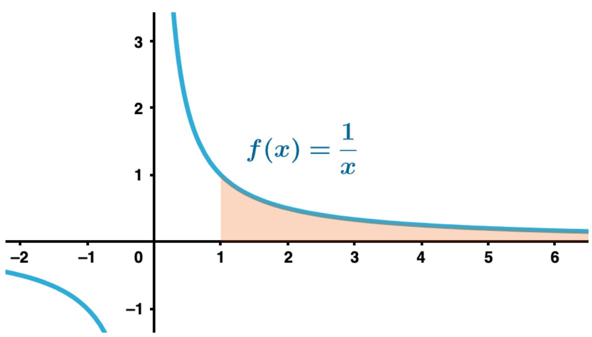
$$\int_a^{\infty} f(x)\,dx = \lim_{t\to\infty} \int_a^t f(x)\,dx$$

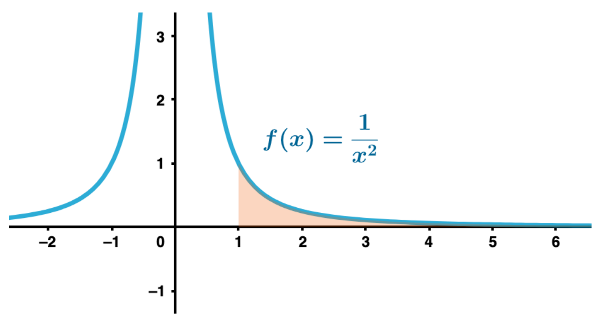
provided this limit exists (as a finite number).

$$\int_1^{\infty} \frac{1}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2}\,dx = \lim_{t\to\infty} \left[\frac{-1}{x}\right]_1^t$$

$$= \lim_{t\to\infty} \left(-\frac{1}{t} + 1\right) = 0 + 1 = 1$$

CONVERGENT!





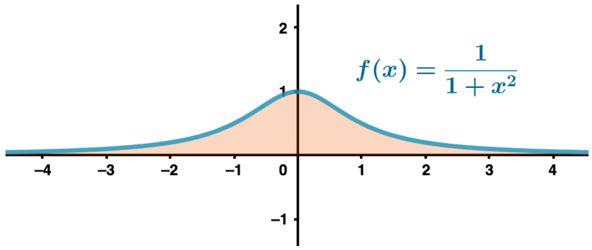
$$\int_1^{\infty} \frac{1}{x}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} \left[\ln|x|\right]_1^t$$

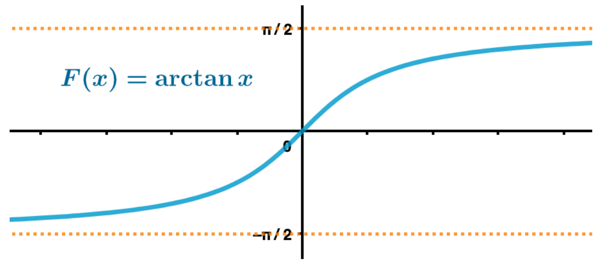
$$= \lim_{t\to\infty} \left(\underbrace{\ln t}_{\to\infty} - \underbrace{\ln 1}_{= 0}\right) = +\infty$$

DIVERGENT (TO $+\infty$)

2 $\displaystyle\int_1^{\infty} \frac{1}{x^p}\,dx$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

p. 571





$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \int_{-\infty}^{0} \frac{1}{1+x^2}\,dx + \int_{0}^{\infty} \frac{1}{1+x^2}\,dx$$

$$\int_{0}^{\infty} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \int_{0}^{t} \frac{1}{1+x^2}\,dx = \lim_{t\to\infty} \Big[\arctan x\Big]_{0}^{t}$$

$$= \lim_{t\to\infty} \Big(\underbrace{\arctan t}_{\to \pi/2} - \underbrace{\arctan 0}_{= 0}\Big) = \frac{\pi}{2} - 0 = \frac{\pi}{2}$$

$$\int_{-\infty}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \int_{t}^{0} \frac{1}{1+x^2}\,dx = \lim_{t\to-\infty} \Big[\arctan x\Big]_{t}^{0}$$

$$= \lim_{t\to-\infty} \Big(\underbrace{\arctan 0}_{= 0} - \underbrace{\arctan t}_{\to -\frac{\pi}{2}}\Big) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

$$\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx = \frac{\pi}{2} + \frac{\pi}{2} = \pi \qquad \text{CONVERGENT}$$