3 Definition of an Improper Integral of Type 2

(a) If f is continuous on $[a, b)$ and is discontinuous at b, then

$$\int_a^b f(x)\,dx = \lim_{t \to b^-} \int_a^t f(x)\,dx$$

if this limit exists (as a finite number).

(b) If f is continuous on $(a, b]$ and is discontinuous at a, then

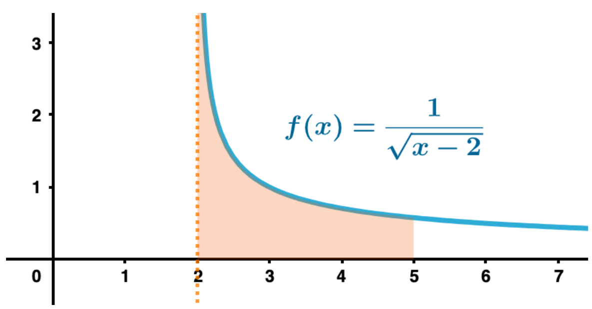
$$\int_a^b f(x)\,dx = \lim_{t \to a^+} \int_t^b f(x)\,dx$$

if this limit exists (as a finite number).

The improper integral $\int_a^b f(x)\,dx$ is called **convergent** if the corresponding limit exists and **divergent** if the limit does not exist.

(c) If f has a discontinuity at c, where $a < c < b$, and both $\int_a^c f(x)\,dx$ and $\int_c^b f(x)\,dx$ are convergent, then we define

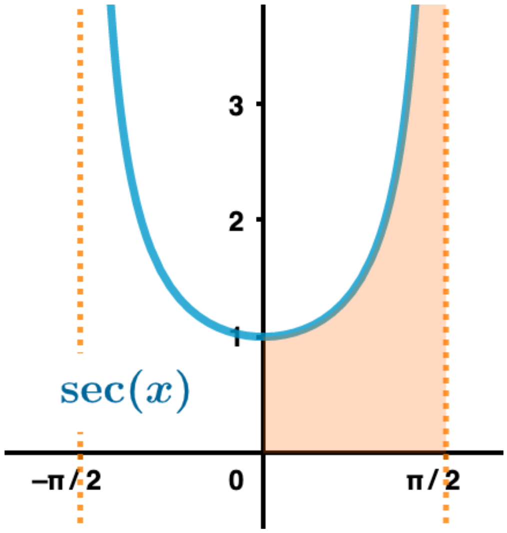
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$$



$$\int_2^5 \frac{1}{\sqrt{x-2}}\,dx = \lim_{t \to 2^+} \int_t^5 \frac{1}{\sqrt{x-2}}\,dx$$

$$= \lim_{t \to 2^+} \Big[2\sqrt{x-2} \Big]_t^5 = \lim_{t \to 2^+} \left(2\sqrt{3} - 2\underbrace{\sqrt{t-2}}_{\to 0} \right) = 2\sqrt{3}$$

CONVERGENT

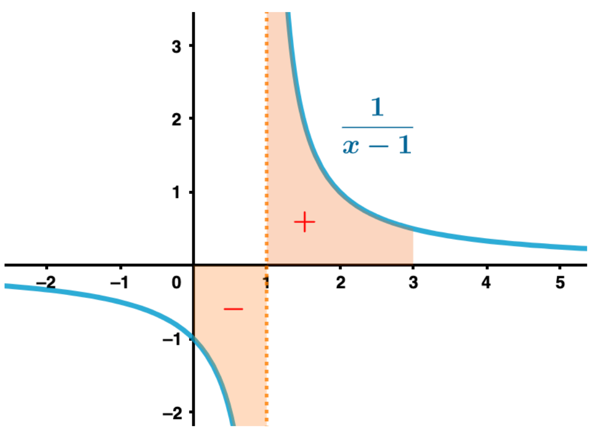


$$\int_0^{\pi/2} \sec x \, dx = \lim_{t \to \frac{\pi}{2}^-} \int_0^t \sec x \, dx$$

$$= \lim_{t \to \left(\frac{\pi}{2}\right)^-} \Big[\ln|\sec x + \tan x| \Big]_0^t$$

$$= \lim_{t \to \left(\frac{\pi}{2}\right)^-} \Big(\underbrace{\ln|\underbrace{\underbrace{\sec t}_{+\infty} + \underbrace{\tan t}_{+\infty}}_{\to +\infty}|}_{\to +\infty} - \underbrace{\ln|1+0|}_{=0} \Big)$$

$$= +\infty \qquad \text{DIVERGENT}$$



$$\int_0^3 \frac{1}{x-1} \, dx$$

$$= \int_0^1 \frac{1}{x-1} \, dx + \int_1^3 \frac{1}{x-1} \, dx$$

$$\int_0^1 \frac{1}{x-1} \, dx = \lim_{t \to 1^-} \int_0^t \frac{1}{x-1} \, dx$$

$$= \lim_{t \to 1^-} \Big[\ln|x-1| \Big]_0^t = \lim_{t \to 1^-} \Big(\underbrace{\ln|\underbrace{t-1}_{\to 0^+}|}_{\to -\infty} - \underbrace{\ln|0-1|}_{=0} \Big) = -\infty$$

DIVERGENT

$$\int_0^1 \frac{1}{x-1}\,dx \quad \text{DIVERGENT}$$

$$\Rightarrow \int_0^3 \frac{1}{x-1}\,dx \quad \text{DIVERGENT}$$

WARNING

$$\int_0^3 \frac{1}{x-1}\,dx \neq \Big[\ln|x-1|\Big]_0^3$$

DIVERGENT!

$$= \ln 2 - \ln 1 = \ln 2$$