MATH 6373
Final Assignment

Below you can find 3 problem sets. Pick one of them and submit complete solutions to all of the problems that it contains. Feel free to structure your solutions as you see fit (for example, it might be helpful to write your solutions as a narrative discussing several problems at once). Problems marked with ⚠ have a higher difficulty.

PROBLEM SET 1 — CHEVALLEY'S THEOREM

The goal of this problem set is to give a proof of Chevalley's theorem.

Definition. Let X be an algebraic variety (or, more generally, a Noetherian topological space). The collection of *constructible subsets* of X is the Boolean algebra of sets generated by the open sets of X. In other words, a subset of X is constructible if it belongs to the smallest family of subsets that contains the open sets of X and is closed under finite intersection and complementation.

Theorem (Chevalley)**.** *Let $\pi\colon X \to Y$ be a morphism of algebraic varieties. Then π sends constructible sets to constructible sets. In particular, the image $\pi(X)$ is a constructible subset of Y.*

In the proof of Chevalley's theorem we will use the following theorem, whose proof we have seen in class.

Theorem (Fundamental Theorem of Elimination Theory)**.** *For any algebraic variety X, the projection $\mathbb{P}^n \times X \to X$ is closed (i.e., it sends closed sets to closed sets).*

Problem 1.1: Show that a set is constructible if and only if it is a finite disjoint union of locally closed sets.

Problem 1.2: Reduce Chevalley's theorem to the case where Y is affine.

Problem 1.3: Reduce further to the case where X is also affine.

Problem 1.4: Reduce further to the case where $X = \mathbb{A}^n \times Y$ and π is the projection onto the second factor.

Problem 1.5: Reduce further to the case where $X = \mathbb{A}^1 \times Y$ and π is the projection onto the second factor.

Problem 1.6: Reduce further to showing that for any affine variety Y and any *locally closed* subset $Z \subseteq \mathbb{A}^1 \times Y$, the image of Z under the projection $\pi\colon \mathbb{A}^1 \times Y \to Y$ is constructible.

Problem 1.7: Reduce further to showing that for any affine variety Y and any *irreducible closed* subset $Z \subseteq \mathbb{A}^1 \times Y$, the image of Z under the projection $\pi\colon \mathbb{A}^1 \times Y \to Y$ is constructible.

After the reduction given in the previous problem, the situation is as follows. The variety Y is affine, so it corresponds to some K-algebra $A(Y) =: R$, and $\mathbb{A}^1 \times Y$ corresponds to the K-algebra $A(\mathbb{A}^1 \times Y) = R[t]$. The projection $\pi\colon \mathbb{A}^1 \times Y \to Y$ corresponds to the inclusion of rings $\pi^\sharp\colon R \hookrightarrow R[t]$.

The irreducible closed subvariety $Z \subseteq \mathbb{A}^1 \times Y$ is cut out by finitely many polynomials $f_1, \ldots, f_m \in R[t]$. Let $a_1, \ldots, a_r \in R$ be the coefficients appearing the polynomials $f_1, \ldots f_m$. Consider the subset $F \subseteq Y$ given by

$$F = \{y \in Y \mid \pi^{-1}(y) \subseteq Z\}.$$

Problem 1.8: Show that F is closed. More precisely, show that $F = V(a_1, \ldots, a_r)$.

Problem 1.9: Reduce Chevalley's theorem to proving the statement of problem 1.7 in the case where Z contains *no* fibers of π.

After the reduction given in problem 1.9, we assume from now on that Z is an irreducible closed subset of $\mathbb{A}^1 \times Y$ containing no fibers of π. We consider the natural inclusion $\mathbb{A}^1 \times Y \subset \mathbb{P}^1 \times Y$ and let $\overline{Z}$ be the closure of Z in $\mathbb{P}^1 \times Y$. Define $Z_\infty = \overline{Z} \setminus Z$. From the fundamental theorem of elimination theory, we know that $\pi(\overline{Z})$ and $\pi(Z_\infty)$ are closed subsets of Y.

Problem 1.10: Show that $\pi(\overline{Z})$ is irreducible.

⚠ **Problem 1.11:** Show that $\pi(Z_\infty) \neq \pi(\overline{Z})$.

Problem 1.12: Using Noetherian induction, conclude the proof of Chevalley's theorem.

Definition. (The right definition of dominant) A rational map $f\colon X \dashrightarrow Y$ is *dominant* if for some open dense set $U \subseteq X$ where f is defined, the image $f(U)$ is dense in Y.

Problem 1.13: Show that in the definition of dominant one can replace the statement "$f(U)$ is dense in Y" with the statement "$f(U)$ contains an open dense subset of Y".

Problem 1.14: Prove that the composition of dominant maps is well-defined and is again dominant.

PROBLEM SET 2 — THE GRASSMANNIAN OF LINES IN 3-SPACE

Consider the following Grassmannian:

$$\mathbb{G} = G(2,4) = \mathbb{G}(1,3) = \{L \subset \mathbb{P}^3 \mid \dim L = 1 \text{ and } L \text{ is linear}\}.$$

We let x_0, x_1, x_2, x_3 be the homogeneous coordinates in $\mathbb{P}^3$, dual to the canonical basis $\{e_0, e_1, e_2, e_3\}$ of K^4. We consider the linear subspaces of K^4 spanned by the last k elements of the canonical basis (as k varies from 1 to 3), and consider the corresponding linear subspaces of $\mathbb{P}^3$:

$$p_0 = \mathbb{P}(Ke_3) \in \mathbb{P}^3, \qquad L_0 = \mathbb{P}(Ke_2 \oplus Ke_3) \subset \mathbb{P}^3, \qquad \Pi_0 = \mathbb{P}(Ke_1 \oplus Ke_2 \oplus Ke_3) \subset \mathbb{P}^3.$$

The sequence $p_0 \in L_0 \subset \Pi_0 \subset \mathbb{P}^3$ is an example of a linear *flag* in $\mathbb{P}^3$.

Recall that the Plücker embedding gives an inclusion $\mathbb{G} \subset \mathbb{P}^5$. The 6 homogeneous coordinates in $\mathbb{P}^5$ are denoted $x_{01}, x_{02}, x_{03}, x_{12}, x_{13}, x_{23}$, and called the Plücker coordinates; they correspond to the 2×2 minors of a 2×4 matrix.

We consider the subset $\Sigma \subset \mathbb{G} \times \mathbb{P}^3$ given by:

$$\Sigma = \{(L, p) \mid p \in L\}.$$

The set Σ is called the *incidence correspondence* of $\mathbb{G}$, or the *universal family* of lines in $\mathbb{P}^3$. The two projections on Σ are denoted $\pi_1 \colon \Sigma \to \mathbb{G}$ and $\pi_2 \colon \Sigma \to \mathbb{P}^3$.

Problem 2.1: Show that $\mathbb{G}$ is cut out in $\mathbb{P}^5$ by the single equation $x_{01}x_{23} - x_{02}x_{13} + x_{12}x_{03} = 0$.

Problem 2.2: Show that Σ is a closed subset of $\mathbb{G} \times \mathbb{P}^5$. Using this, show that for any closed subvarieties $X \subseteq \mathbb{P}^3$ and $\mathcal{Y} \subseteq \mathbb{G}$, the sets $\mathcal{X} = \{L \mid L \cap X \neq \emptyset\} \subseteq \mathbb{G}$ and $Y = \cup_{L \in \mathcal{Y}} L \subseteq \mathbb{P}^3$ are also closed subvarieties.

Problem 2.3: For any point $p \in \mathbb{P}^3$ and plane $\Pi \subset \mathbb{P}^3$ containing p, let $\Sigma_{p,\Pi} \subset \mathbb{G}$ be the locus of lines containing p and contained in Π. Show that under the Plücker embedding, $\Sigma_{p,\Pi}$ is mapped to a line, and that conversely every line in $\mathbb{P}^5$ contained in $\mathbb{G}$ is of the form $\Sigma_{p,\Pi}$ for some p and Π.

Problem 2.4: For any point $p \in \mathbb{P}^3$, let $\Sigma_p \subset \mathbb{G}$ be the locus of lines containing p; for any plane $\Pi \subset \mathbb{P}^3$, let $\Sigma_\Pi \subset \mathbb{G}$ be the locus of lines contained in Π. Show that under the Plücker embedding, both Σ_p and Σ_Π get mapped into two-planes in $\mathbb{P}^5$, and that conversely every two-plane $\Lambda \simeq \mathbb{P}^2 \subset \mathbb{G} \subset \mathbb{P}^5$ is of the form Σ_p for some p or of the form Σ_Π for some Π.

Problem 2.5: For any line $L \subset \mathbb{P}^3$, let Σ_L be the locus of lines intersecting L. Show that $\Sigma_L = \mathcal{H} \cap \mathbb{G}$, where $\mathcal{H} \subset \mathbb{P}^5$ is a hyperplane. Show that Σ_L is isomorphic to a projective cone over $\mathbb{P}^1 \times \mathbb{P}^1$. Given any hyperplane $\mathcal{H} \subset \mathbb{P}^5$, is it true that the intersection $\mathcal{H} \cap \mathbb{G}$ is of the form Σ_L for some L?

The varieties

$$\{L_0\} \qquad \Sigma_{p_0,\Pi_0} \qquad \Sigma_{p_0} \qquad \Sigma_{\Pi_0} \qquad \Sigma_{L_0} \qquad \mathbb{G}$$

are known as the *Schubert varieties* of $\mathbb{G}$. The collection of Schubert varieties under inclusion forms a partially ordered set, known as the *Bruhat poset* of $\mathbb{G}$. Given a Schubert variety $\Omega \subseteq \mathbb{G}$, we define $\Omega^\circ = \Omega \smallsetminus \cup_{\Omega' \subset \Omega} \Omega'$. The Ω° are known as the *Schubert cells* of $\mathbb{G}$, and $\mathbb{G}^\circ$ is known as the *big cell*. Notice that the Schubert cells give a partition of $\mathbb{G}$.

Problem 2.6: Draw the Hasse diagram for the Bruhat poset of $\mathbb{G}$.

Problem 2.7: Give the equations of all the Schubert varieties in $\mathbb{G}$.

Problem 2.8: Each point in G can be represented by a 2×4 matrix. Any such matrix is row-equivalent to a unique matrix in reduced row echelon form. The columns corresponding to the leading 1's in the reduced row echelon form are called the *pivots* of the matrix. Show that two matrices belong to the same Schubert cell if and only if they have the same pivots. In particular, show that each Schubert cell is of the form $\mathbb{A}^d$ for some d.

Problem 2.9: Let $L_1, L_2 \subset \mathbb{P}^3$ be skew lines (i.e, $L_1 \cap L_2 = \emptyset$). Let $\mathcal{Q} \subset \mathbb{G}$ be the locus of lines meeting both L_1 and L_2. Show that $\mathcal{Q} \simeq \mathbb{P}^1 \times \mathbb{P}^1$. What happens if L_1 and L_2 meet?

⚠ **Problem 2.10:** Let $Q \subset \mathbb{P}^3$ be a smooth quadric surface, and let $\mathcal{Q} \subset \mathbb{G}$ be the locus of lines contained in Q. Show that $\mathcal{Q}$ has two connected components, $\mathcal{Q} = \mathcal{C}_1 \cup \mathcal{C}_2$, corresponding to two plane conic curves $\mathcal{C}_1 \subset \Lambda_1 \subset \mathbb{P}^5$ and $\mathcal{C}_2 \subset \Lambda_2 \subset \mathbb{P}^5$, where Λ_1 and Λ_2 are two complementary two-planes in $\mathbb{P}^5$. Conversely, show that if $\mathcal{C} \subset \Lambda \subset \mathbb{P}^5$, where Λ is a two-plane not contained in $\mathbb{G}$, $\mathcal{C}$ is a smooth conic in Λ, and $\mathcal{C} \subset \mathbb{G}$, then $Q = \cup_{L \in \mathcal{C}} L \subset \mathbb{P}^3$ is a smooth quadric surface.

Optional Problem (for those familiar with (co)homology). Using the Schubert cell decomposition, compute the (co)homology groups of $\mathbb{G}$ when the ground field is $\mathbb{C}$.

PROBLEM SET 3 — SOME BIRATIONAL GEOMETRY

Consider the rational map from $\mathbb{P}^2$ to $\mathbb{P}^4$ given by

$$\varphi(x_0 : x_1 : x_2) = (x_0^2 : x_0x_1 : x_1^2 : x_0x_2 : x_1x_2).$$

The closure of the image of φ will be denoted by $X \subset \mathbb{P}^4$ and is called the *cubic scroll* in $\mathbb{P}^4$.

Problem 3.1: Show that X is the closure of the image of the Veronese surface $S \subset \mathbb{P}^5$ under a projection $\pi_p \colon \mathbb{P}^5 \dashrightarrow \mathbb{P}^4$ from a point $p \in \mathbb{P}^5$. Find the point of projection p.

Problem 3.2: Show that X is isomorphic to the blow-up of the plane $\mathbb{P}^2$ at the point $q = (0 : 0 : 1)$.

Via the identification of X with the blow-up of $\mathbb{P}^2$, the exceptional divisor of the blow-up corresponds to a curve $E \subset X \subset \mathbb{P}^4$. This curve E is called the *directrix* of X.

Problem 3.3: Show that the directrix E is a line in $\mathbb{P}^4$.

Problem 3.4: For each line L in $\mathbb{P}^2$ containing the point q, the strict transform $\widetilde{L}$ of L in the blow-up corresponds to a curve $\widetilde{L} \subset X \subset \mathbb{P}^4$. Show that $\widetilde{L}$ is a line in $\mathbb{P}^4$ intersecting E in one point.

Problem 3.5: For each line C in $\mathbb{P}^2$ *not* containing the point q, the strict transform $\widetilde{C}$ of C in the blow-up corresponds to a curve $\widetilde{C} \subset X \subset \mathbb{P}^4$. Show that $\widetilde{C}$ is a plane conic in $\mathbb{P}^4$ (i.e, $\widetilde{C}$ is contained in a two-plane $\Pi \simeq \mathbb{P}^2 \subset \mathbb{P}^4$ and $\widetilde{C}$ is a conic in Π). Show that $\widetilde{C}$ is disjoint from E.

Problem 3.6: Let $r \in X$ be any point, and consider the projection $\pi_r \colon X \dashrightarrow \mathbb{P}^3$. Let $Q \subset \mathbb{P}^3$ be the closure of the image of π_r. Show that Q is a quadric surface, and compute the rank of Q in terms of the point r.

Problem 3.7: Let $Q \subset \mathbb{P}^n$ be a quadric hypersurface, consider a point $p \in Q$, and the associated projection $\pi_p \colon Q \dashrightarrow \mathbb{P}^{n-1}$. When is π_p birational?

⚠ **Problem 3.8:** In the setup of the previous problem, assume that Q has full rank. Describe π_p in terms of blow-ups and blow-downs (a blow-down is the inverse of a blow-up).

In class we saw that the blow-up of a variety X along one function $f \in A(X)$ gives an isomorphism. We also saw that the blow-up of X along $f_1, \dots, f_r \in A(X)$ only depends on the ideal $(f_1, \dots, f_r)$. Since the ideal (f) gives a subvariety of X of codimension one, we can ask if the blow-up along subvarieties of codimension one is always an isomorphism. This is certainly not the case, as one can easily see in the case of blow-ups of singular curves along points. Below are two examples in dimension two.

Problem 3.9: Consider the surface $S \subset \mathbb{A}^3$ given by the equation $x^2 = y^2 z$. This surface is known as the *Whitney umbrella* (the name "umbrella" comes from its picture when the ground field is $\mathbb{R}$). Consider the line $L \subset S \subset \mathbb{A}^3$ given by $x = y = 0$ (i.e., L is the z-axis). Compute and describe the blow-up of S along L, and determine whether S and the blow-up are isomorphic.

Problem 3.10: Consider the surface $S \subset \mathbb{A}^3$ given by the equation $x^2 = yz$. This surface is known as a *quadric cone* (notice that it is isomorphic to the surface given by the equation $z^2 = x^2 + y^2$). Consider the line $L \subset S \subset \mathbb{A}^3$ given by $x = y = 0$. Compute and describe the blow-up of S along L, and determine whether S and the blow-up are isomorphic.